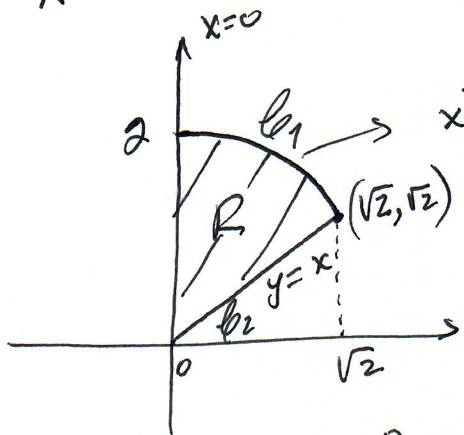
15.10 CHANGE OF VARIABLES IN MULTIPLE INTEGRALS

15.3 # 8)

$$I = \iint_R (2x-y)\, dA,$$ where R is region in 1st quadrant



$$I = \int_{0}^{\sqrt{2}} \int_{x}^{+\sqrt{4-x^2}} (2x-y)\, dA$$
$x \qquad y$

Changing to polar Coords.

$$\int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=0}^{r=2} (2r\cos\theta - r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left[\frac{2}{3} r^3 \cos\theta - \frac{r^3}{3}\sin\theta\right]\Bigg|_0^2 d\theta = \int_{\pi/4}^{\pi/2} \left[\frac{2}{3}(8)\cos\theta - \frac{8}{3}\sin\theta\right] d\theta$$

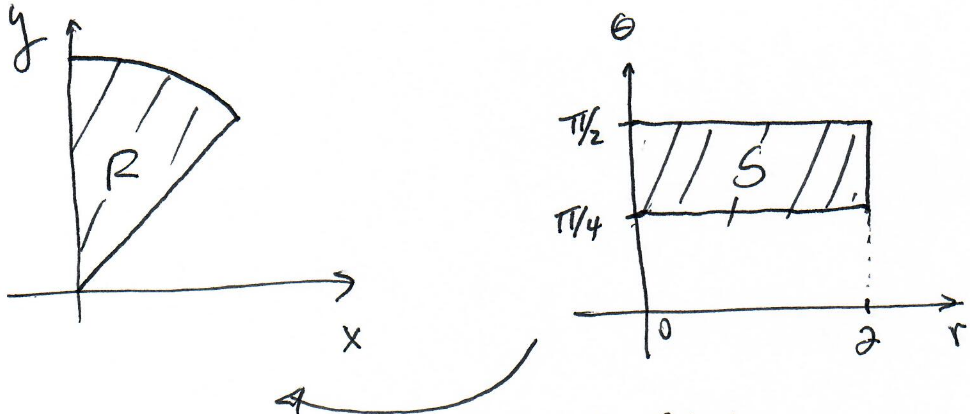
$$= \frac{16}{3}\sin\theta\Bigg|_{\pi/4}^{\pi/2} + \frac{8}{3}\cos\theta\Bigg|_{\pi/4}^{\pi/2} = \frac{16}{3}\left[\sin\pi/2 - \sin\pi/4\right]$$

$$+ \frac{8}{3}\left[\cos\frac{\pi}{2} - \cos\frac{\pi}{4}\right] = \frac{16}{3}\left(1 - \frac{\sqrt{2}}{2}\right) + \frac{8}{3}\left(-\frac{\sqrt{2}}{2}\right)$$

$$= \frac{16}{3} - \frac{24}{3}\left(\frac{\sqrt{2}}{2}\right) = \frac{16}{3} - 4\sqrt{2}.$$

This is a particular case of

$$\iint_R f(x,y)\,dA = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=a}^{b} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$



$$T: \begin{aligned} x(r,\theta) &= r\cos\theta \\ y(r,\theta) &= r\sin\theta \end{aligned}$$

$$\int_{x=0}^{\sqrt{2}}\int_{y=x}^{y=\sqrt{4-x^2}} (2x-y)\,dy\,dx = \int_{\theta=\pi/4}^{\theta=\pi/2}\int_{r=0}^{r=2} (2r\cos\theta - r\sin\theta)\, r\,dr\,d\theta$$

Notice that

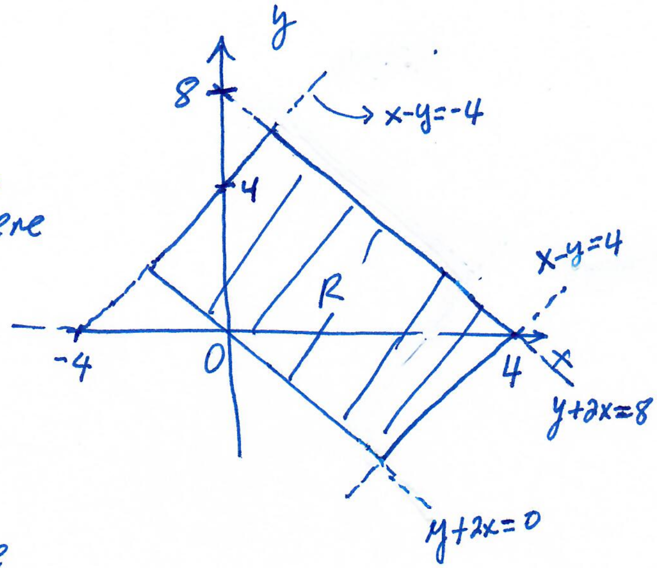
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \text{Jacobian of Transformation } T = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

15.9 Change of Variables for Multiple Integrals.

Want to compute

$$I = \iint_R (4x+8y)\,dA, \quad \text{where}$$

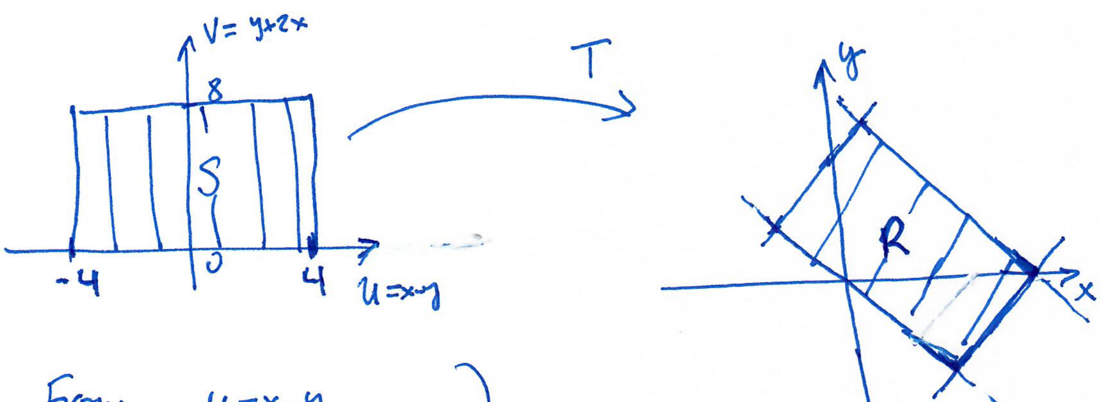


<u>Step I:</u> Find an appropriate transformation

$$T: S \longrightarrow R,$$ where S is a convenient rectangular region.

$$(u,v) \longrightarrow (x,y), \quad x = x(u,v), \quad y = y(u,v).$$

A good choice is $u = x-y$, $v = y+2x$.



From
$$\begin{aligned} u &= x-y \\ v &= y+2x \\ \hline u+v &= 3x \end{aligned}$$

$\Rightarrow x = \dfrac{u+v}{3} \Rightarrow y = x-u$

$y = \dfrac{u}{3} + \dfrac{v}{3} - u = \dfrac{v-2u}{3}$

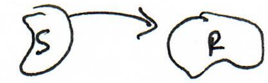
$$\Rightarrow \quad x = x(u,v) = \frac{u+v}{3}$$

$$y = y(u,v) = \frac{v-2u}{3}.$$

15.9 Change of Vabs. Multiple Integrals

Theorm 9.- Consider a transf.

$$T: S \subset \mathbb{R}^2 \to R \subset \mathbb{R}^2$$

T: 

$$(u,v) \longrightarrow \begin{array}{l} x = x(u,v) \\ y = y(u,v) \end{array}$$

① T is C' conts. differentiable $\longrightarrow$ (It means $x(u,v)$ and $y(u,v)$ have conts 1st partial derivatives)

② The Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$

is always of the same sign (never zero). on the interior of S.

then, T is one-to-one. on the interior of S.

③ R is bounded by a simple closed curve ∂R (It may be piecewise smooth).

④ f is conts on R.

Then,

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv.$$

Step 2: Apply theorem 9

$$J(u,v) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{vmatrix} = \frac{1}{9} + \frac{2}{9} = \frac{3}{9} = \frac{1}{3}$$

Therefore,

$$I = \iint_R (4x+8y)\,dA = \iint_S \left[4\left(\frac{u+v}{3}\right) + 8\left(\frac{v-2u}{3}\right)\right] dA_s$$

$$= 4\int_{-4}^{4}\int_{0}^{8} \left[\frac{u}{3} + \frac{v}{3} + \frac{2v}{3} - \frac{4u}{3}\right] \overbrace{\left|\left(\frac{1}{3}\right)\right|}^{=|J(u,v)|} dv\,du$$

$$u \qquad v$$

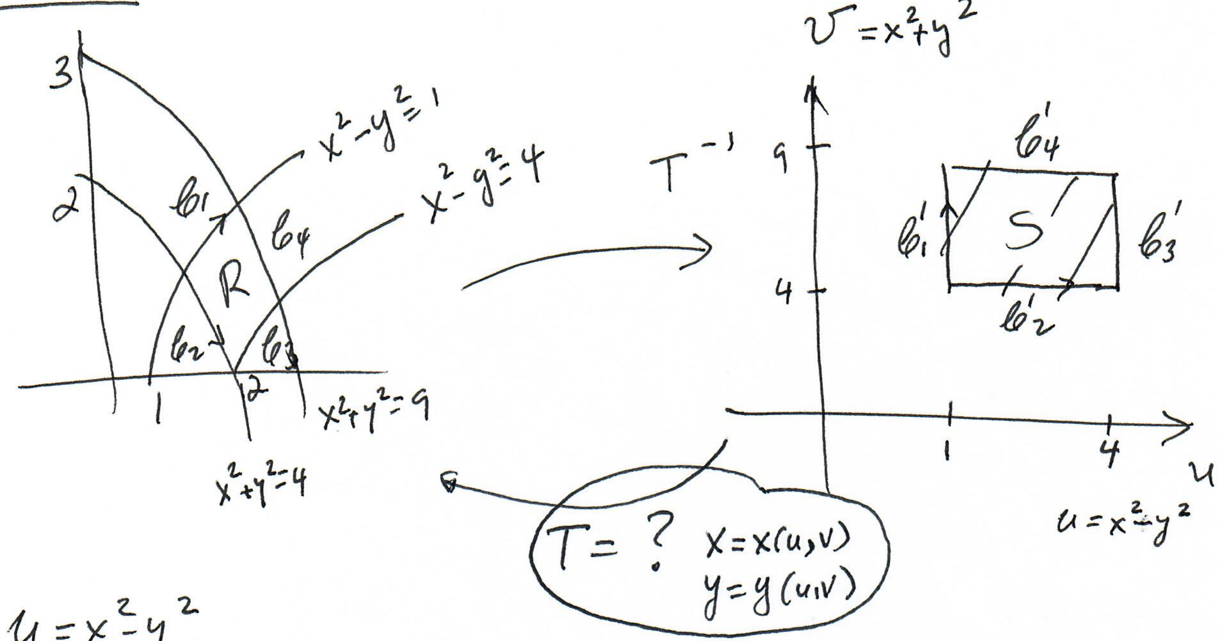
$$= \frac{4}{3}\int_{-4}^{4}\int_{0}^{8} (v-u)\,dv\,du = \frac{4}{3}\int_{-4}^{4} \left(\frac{v^2}{2} - uv\right)\Big|_0^8 du$$

$$= \frac{4}{3}\int_{-4}^{4} (32-8u)\,du = \frac{32}{3}\int_{-4}^{4} (4-u)\,du = \frac{32}{3}\left(4u - \frac{u^2}{2}\right)\Big|_{-4}^{4}$$

$$= \frac{32}{3}\left(16 - \cancel{8} - (-16 - \cancel{8})\right) = \frac{32}{3}(32)$$

$$= \frac{1024}{3}$$

Example 2.- Evaluate $I = \iint_R xy\,dx\,dy$



$$u = x^2 - y^2$$
$$v = x^2 + y^2$$
$$u + v = 2x^2 \Rightarrow \boxed{x^2 = \frac{u+v}{2}} \qquad \boxed{y^2 = \frac{v-u}{2}}$$

Also

$$J(u,v) = \begin{vmatrix} \frac{\partial}{\partial u}\left(\frac{u+v}{2}\right)^{1/2} & \frac{\partial}{\partial u}\left(\frac{v-u}{2}\right)^{1/2} \\ \frac{\partial}{\partial v}\left(\frac{u+v}{2}\right)^{1/2} & \frac{\partial}{\partial v}\left(\frac{v-u}{2}\right)^{1/2} \end{vmatrix} =$$

$$= \begin{vmatrix} \frac{1}{2}\left(\frac{u+v}{2}\right)^{-1/2}\frac{1}{2} & +\frac{1}{2}\left(\frac{v-u}{2}\right)^{-1/2}\left(-\frac{1}{2}\right) \\ \frac{1}{2}\left(\frac{u+v}{2}\right)^{-1/2}\frac{1}{2} & \frac{1}{2}\left(\frac{v-u}{2}\right)^{-1/2}\frac{1}{2} \end{vmatrix} = \begin{vmatrix} \frac{1}{4}\left(\frac{2}{u+v}\right)^{1/2} & -\frac{1}{4}\left(\frac{2}{v-u}\right)^{1/2} \\ \frac{1}{4}\left(\frac{2}{u+v}\right)^{1/2} & \frac{1}{4}\left(\frac{2}{v-u}\right)^{1/2} \end{vmatrix} =$$

$$= \frac{1}{16}\frac{2}{(v^2-u^2)^{1/2}} + \frac{1}{16}\frac{2}{(v^2-u^2)^{1/2}} = \frac{1}{4}\frac{1}{\sqrt{v^2-u^2}}$$

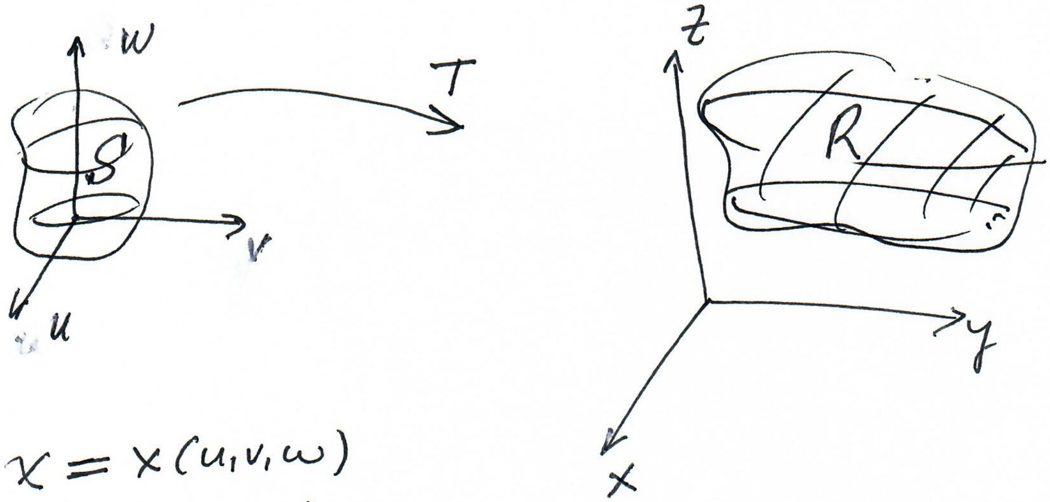
$$\therefore I = \int_{\substack{1\\u}}^{4}\int_{\substack{4\\v}}^{9} \cancel{\left(\frac{u+v}{2}\right)^{1/2}}\left(\frac{v-u}{2}\right)^{1/2}\frac{1}{4}\frac{1}{\cancel{(v^2-u^2)^{1/2}}}\,dv\,du = \frac{1}{8}\iint_S dv\,du =$$

$$= \frac{1}{8}\int_1^4\int_4^9 dv\,du = \frac{1}{8}\,15 = \frac{15}{8}$$

Triple Integrals

Let

$$T : D \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$



$$x = x(u,v,w)$$
$$y = y(u,v,w)$$
$$z = z(u,v,w)$$

Let's call the Jacobian of T

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Under same hypothesis to those of thm 9, the change of Variables for triple integrals satisfies

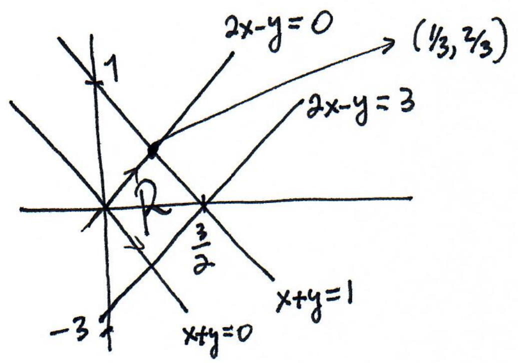
$$\boxed{\iiint_R f(x,y,z)\, dV = \iiint_S f\left(x(u,v,w), y(u,v,w), z(u,v,w)\right) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw.}$$

Example 1

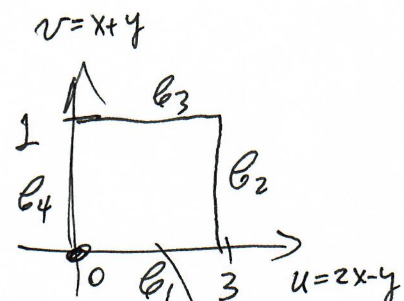
Evaluate the integral $\iint_R (x+y)^2\,dx\,dy$

where the region R is given by

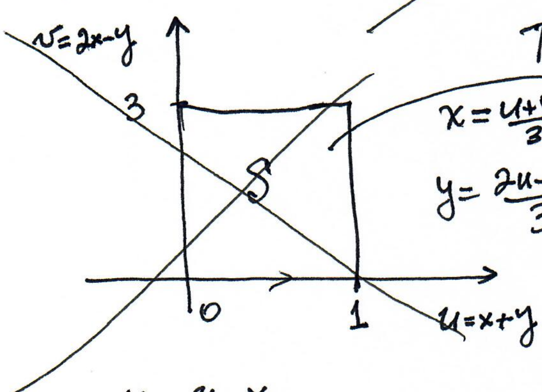
Another example



$2x-1+x=0$
$3x=1 \quad x=1/3$



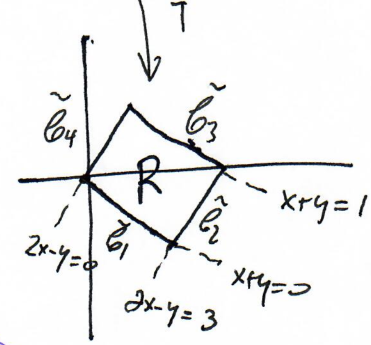
Transformation T



T

$x = \frac{u+v}{3}$

$y = \frac{2u-v}{3}$



$y = u - x$

$\Rightarrow \quad v = 2x - u + x = 3x - u$

$\Rightarrow \boxed{x = \frac{u+v}{3}} \Rightarrow y = u - \frac{u+v}{3} = \frac{2u-v}{3} \Rightarrow \boxed{y = \frac{2u-v}{3}}$

$$\iint_R (x+y)^2\,dx\,dy = \int_{0}^{1}\int_{0}^{3}\left[\frac{u+v}{3} + \frac{2u-v}{3}\right]^2 |J(u,v)|\,du\,dv$$

$\qquad\qquad u \qquad v \qquad \left(\frac{3u}{3}\right)^2 = u^2$

Now,

$$J(u,v) = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}$$

$$\Rightarrow I = \int_0^1\int_0^3 u^2 \frac{1}{3}\,dv\,du = \int_0^1 \frac{1}{3} u^2 v \Big|_0^3 du = \int_0^1 \frac{1}{\cancel{3}} u^2 \cdot \cancel{3}\,du = \frac{u^3}{3}\Big|_0^1 = \frac{1}{3}$$